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On the form factors of relevant operators and their cluster property

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Abstract. We compute the form factors of the relevant scaling operators in a class of integrable models without internal symmetries by exploiting their cluster properties. Their identification is established by computing the corresponding anomalous dimensions by means of the Delfino–Simonetti–Cardy sum rule and further confirmed it by comparing some universal ratios of the nearby non-integrable quantum field theories with their independent numerical determination.

1. Introduction

In this work we present a detailed investigation of the matrix elements†

$$F^{\mathcal{O}}_{a_1,a_2,\ldots,a_n}(\theta_1,\ldots,\theta_n) = \langle 0|\mathcal{O}(0)|A_{a_1}(\theta_1)\ldots A_{a_n}(\theta_n)\rangle \tag{1.1}$$

(the so-called form factors (FFs)) in a class of integrable two-dimensional quantum field theories. Our specific aim is to check some new theoretical ideas which concern the relationships between three different regimes which two-dimensional quantum field theories may have, namely the ones ruled by conformal invariance, integrable or non-integrable dynamics.

Conformal field theories (CFT) and the associated off-critical integrable models (IM) have been extensively studied in recent years: as a result of these analyses a great deal of information has been obtained particularly on correlation functions of a large number of statistical mechanical models in their scaling limit and on physical quantities related to them (see for instance [1–8]). In this context, a crucial problem often consists of the determination of the spectrum of the scaling operators away from criticality, namely their correct identification by means of the set of their FFs. This is one of the issues addressed in this work.

FFs also play a crucial role in estimating non-integrable effects. Let us first recall that the above CFT and IM regimes cannot obviously exhaust all possible behaviours that statistical models and quantum field theories can have since typically they do not possess an infinite number of conservation laws. This means that in general we have to face all kinds of phenomena and complications associated to non-integrable models (NIM). The scattering amplitudes and the matrix elements of the quantum operators will have in these cases a pattern of analytic singularities due both to the presence of higher thresholds and to

† The rapidity variables θ_i conveniently parametrize the dispersion relations of the particles, $E_i = m_i \cosh\theta_i$, $p_i = m_i \sinh\theta_i$.

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the appearance of resonances. A first step forward in their analysis has recently been taken in [9] where it has been shown that some interesting examples of NIM may be obtained as deformations of integrable models. The action of such theories can correspondingly be written as

$$\mathcal{A} = \mathcal{A}_{\rm int} + \sum_i \lambda_i \int \mathrm{d}^2x\, \Psi_i(x) \tag{1.2}$$

$\mathcal{A}_{\rm int}$ being the action of the integrable model. Since the exact expressions (1.1) of the FFs of the integrable theories are all assumed calculable, in particular, the ones of the fields $\Psi_i(x)$ entering equation (1.2), one is inclined to study the non-integrable effects by using the Born series based on the FFs. Although at first sight this still remains a difficult task (and generally, it is indeed so), there may be favourable circumstances where the analysis simplifies considerably. For instance, as long as there is only a soft breaking of integrability, it has been shown in [9] that the complications of the higher terms in the series can often be avoided since the most relevant corrections only come from the lowest approximation. If this is the case, one can extract an important amount of information with relatively little effort: a significant set of physical quantities to look at is provided for instance by universal ratios, such as the ones relative to the variations of the masses or of the vacuum energy density $\mathcal{E}_{\rm vac}$: if the breaking of integrability is realized by means of a single field $\Psi(x)$, those are expressed by

$$\begin{aligned} \frac{\delta m_i}{\delta m_j} &= \frac{m_j^{(0)}}{m_i^{(0)}} \frac{F_{ii}^{\Psi}(\mathrm{i}\pi)}{F_{jj}^{\Psi}(\mathrm{i}\pi)} \\ \frac{\delta \mathcal{E}_{\rm vac}}{m_1^{(0)}\delta m_1} &= \frac{\langle 0|\Psi|0\rangle}{F_{11}^{\Psi}(\mathrm{i}\pi)} \end{aligned} \tag{1.3}$$

where $m_i^{(0)}$ refers to the (unperturbed) mass spectrum of the original integrable theory. It is thus evident that also to estimate the non-integrable effects associated to a given operator $\Psi(x)$ one must face the problem of correctly identifying its FFs.

Two new results on the relationship between CFT and IM have recently been derived by Delfino *et al* [10]. The first result consists of a new sum rule which relates the conformal dimension Δ^{ϕ} of the operator $\phi(x)$ to the off-critical (connected) correlator $\langle\Theta(x)\phi(0)\rangle_c$, where $\Theta(x)$ is the trace of the stress–energy tensor†

$$\Delta^{\phi} = -\frac{1}{4\pi\langle\phi\rangle} \int \mathrm{d}^2x\, \langle\Theta(x)\phi(0)\rangle_c. \tag{1.4}$$

This sum rule is closely related to the analogous expression for the conformal central charge, c [11]

$$c = \frac{3}{4\pi} \int \mathrm{d}^2x\, |x|^2 \langle\Theta(x)\Theta(0)\rangle_c. \tag{1.5}$$

Equations (1.4) and (1.5) express elegant relationships between conformal and off-critical data, but more importantly, they provide very concrete and efficient tools to characterize the scaling limit of the off-critical models.

As for the second result, it has been suggested by the aforementioned authors of [10], that the FFs of the relevant scaling fields‡ of an integrable field theory—in absence of

† The sum rule in the form of equation (1.4) may be violated by effect of renormalization of the operators outside the critical point, as clarified in the original reference [10]. This is however not the case for the field theories and the operators considered in this work.

‡ Hereafter we shall use the short expression 'scaling fields' to actually denote the off-critical operators which reduce to the scaling fields in the conformal limit.

internal symmetries—are in one-to-one correspondence with the independent solutions of the so-called *cluster equations*

$$\lim_{\Lambda\to\infty} F^{\Phi}_{a_1,a_2,\ldots,a_k,a_{k+1},\ldots,a_{k+l}}(\theta_1,\theta_2,\ldots,\theta_k,\Lambda+\theta_{k+1},\ldots,\Lambda+\theta_{k+l}) = \frac{1}{\langle\Phi\rangle} F^{\Phi}_{a_1,a_2,\ldots,a_k}(\theta_1,\theta_2,\ldots,\theta_k) F^{\Phi}_{a_{k+1},\ldots,a_{k+l}}(\theta_{k+1},\ldots,\theta_{k+l}). \tag{1.6}$$

The above limit is supposed to test the ultraviolet regime of the theory since the shift by Λ changes the relative energies of the two subsets of particles. In this limit chiral splitting is expected to occur. Equations (1.6) can be imposed on the FFs *in addition* to the customary functional and residue equations which they satisfy (see in this respect also [1, 12]). If this *cluster hypothesis* is valid, we would have a clear method to identify the matrix elements of all the relevant operators, at least in the case of theories without symmetries. It must be stressed that until now this task has often been a matter of keen guess work and mostly based on physical intuition.

It turns out that a check of the above *cluster hypothesis* provides a well suited forum for testing several theoretical aspects. In fact, the most direct way of confirming the above idea is first to solve the general functional equations of the FFs with the additional constraints of the cluster equations (1.6) and to see whether *the number of independent solutions equals the number of relevant fields* in the corresponding Kac table. If the above check turns out to be positive, one may use sum rule (1.4) in order to achieve the correct identification of the (supposed) primary relevant operators ϕ_i: from the values of the partial sums one can in fact infer the value of the anomalous dimension and correspondingly recognize the operator. Additional confirmation may also come from the employment of equations (1.3) relative to non-integrable field theories. In fact, one can regard the primary field $\phi_i(x)$ under investigation as that operator which spoils the integrability of the original theory and therefore compare predictions (1.3) based on its FFs with their independent numerical determinations which may be obtained by means of the truncation method [13]. Note that a successful test of this kind could also be interpreted the other way around, namely as a further proof of the effectiveness of formulae (1.3) in estimating non-integrable effects.

The models on which we have chosen to test the above considerations are integrable deformations of the first representatives of the non-unitary conformal series† $\mathcal{M}(2, 2n+1)$, $n \geqslant 2$. They belong to the class of universality of solvable RSOS lattice models *à la* Andrews–Baxter–Forrester although with negative Boltzmann weights [14, 15]: their simplest example is given by the quantum field theory associated to the so-called Yang–Lee model which describes the distribution of zeros in the grand canonical partition function of the Ising model in a complex magnetic field [16, 17]. These models do not have any internal symmetry and all their fields are relevant operators: hence, they are ideal for our purposes. Moreover, the nature of their massive and conformal phases is simple enough. The consequence for their relative simplicity is, however, the presence of typical non-unitary phenomena, as imaginary coupling constants or negative values of the anomalous dimensions and central charge, together with the anomalous poles in the S-matrix which induce an unusual analytic structure in the FFs [4, 17, 18].

† The conformal weights and central charge are given, respectively, by

$$\Delta_{1,a} = -\frac{(a-1)(2n-a)}{2(2n+1)} \qquad a = 1, 2, \ldots, 2n$$

$$c = -\frac{2(6n^2-7n+1)}{2n+1} \qquad n = 2, 3, \ldots.$$

The paper is organized as follows. In section 2 we discuss the general strategy which can be employed in order to compute the FFs of the relevant operators in the integrable deformations of the models $\mathcal{M}(2, 2n+1)$. In sections 3 and 4 we present a detailed analysis of the FFs of the models $\mathcal{M}(2, 7)$ and $\mathcal{M}(2, 9)$, which are the first non-trivial examples on which to check all the theoretical ideas discussed above. In fact, for the first model, $\mathcal{M}(2, 5)$, the *cluster hypothesis* is easily verified: the only solution of the FF equations is the sequence of functions determined in [7] which indeed fulfil the cluster equations (1.6) and are easily identified with the matrix elements of the only relevant field of the Yang–Lee model. The two models $\mathcal{M}(2, 7)$ and $\mathcal{M}(2, 9)$ represent, somehow, the best playground for our purposes because they give rise to integrable models under each of their possible (individual) deformations and also because they optimize the size of the lengthy numerical output which we present for the solutions of the nonlinear equations. Moreover, although there is, in principle, no obstacle to extend the analysis to all the models $\mathcal{M}(2, 2n+1)$, these are the simplest cases from a computational point of view since the larger the value of the index n the higher the order of the system of algebraic equations to be solved to determine the FFs. Finally, our conclusions are given in section 5. Two appendices complete the paper: appendix A gathers all important formulae relative to the parametrization of the two-particle FFs and appendix B collects the S-matrices of the models analysed.

2. Outline of our strategy

In this section we discuss the general strategy needed in order to obtain the FFs of the scaling primary fields of the integrable deformations $\phi_{1,k}$ of the conformal models $\mathcal{M}(2, 2n+1)$ (hereafter denoted by the shorthand notation $[\mathcal{M}(2, 2n+1)]_{(1,k)}$). The deforming field $\phi_{1,k}$ can be one of the operators $\phi_{1,2}$, $\phi_{1,3}$ or possibly some other primary field which gives rise to an integrable deformation.

The starting point in the computation of the FFs is the correct parametrization of the two-particle ones which is given in detail in appendix A. This is a non-trivial task in the case of non-unitary models for the reason that the exact S-matrices of these models are usually plagued by a plethora of anomalous poles [18]. By this we mean, for example, simple poles which are not related to any bound state, or, more generally, any poles which apparently do not have the standard diagrammatic interpretation of [19]. Consider, for example, the S-matrices listed in the tables of appendix B relative to the integrable deformations of the models $\mathcal{M}(2, 7)$ and $\mathcal{M}(2, 9)$ where the anomalous poles have been labelled with $\mathcal{B}$, $\mathcal{D}$ or $*$. The origin of these poles may be explained according to the ideas put forward in [20]. In particular, poles of type $\mathcal{B}$ and $\mathcal{D}$ are due to multiparticle processes of the kind described respectively by the 'butterfly' and 'dragonfly' diagrams drawn in figures 2 and 3 respectively. These multiloop processes induce in the S-matrix simple poles rather than higher-order ones because the internal lines of these diagrams cross at relative rapidity values relative to some zeros of their corresponding two-particle S-matrix element: this gives rise to a partial cancellation of the poles.

The adopted parametrization for two-particle FFs is directly related to the pole structure of the S-matrix. This yields expression (A.4) whose functional form is set except for the coefficients $a^{(k)}_{ab,\Phi}$ appearing in expansion (A.8) of the polynomials $Q^{\Phi}_{ab}(\theta)$. The degree $k^{\max}_{ab,\Phi}$ of these polynomials is fixed by the asymptotic behaviour of the FFs for large rapidities which depends, of course, on the field Φ [3]. For the case of two-particle FFs of cluster operators, it is easy to see that they are subject to have for large θ at most a constant limit†.

† The limit may vanish in the presence of symmetries.

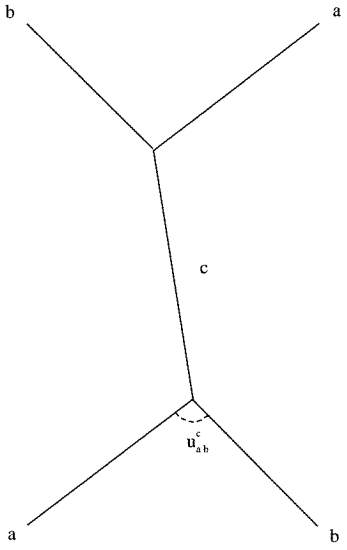

Figure 1. Bound-state simple pole diagram.

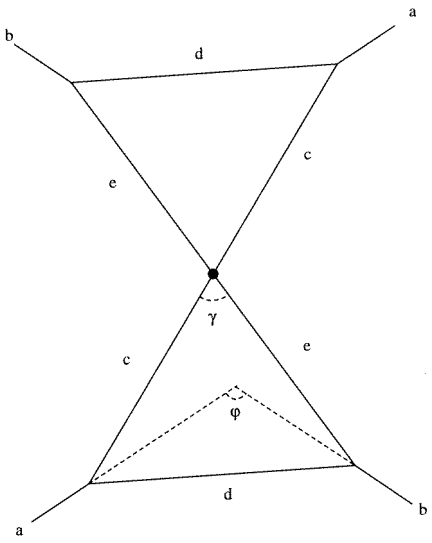

Figure 2. 'Butterfly' diagram.

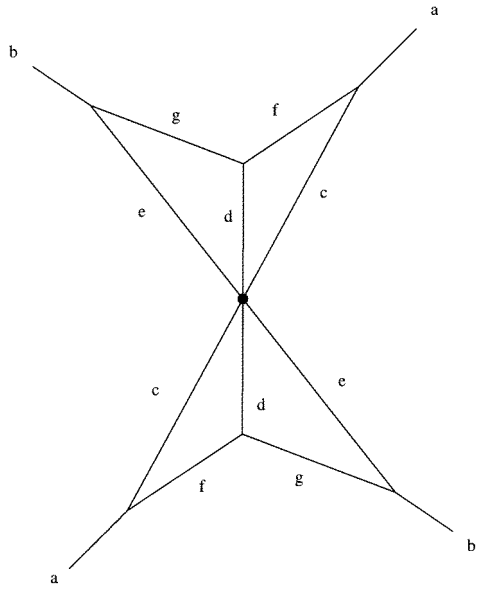

Figure 3. 'Dragonfly' diagram.

In fact, for two-particle FFs equations (1.6) read

$$\lim_{\theta\to\infty} F^{\Phi}_{ab}(\theta) = F^{\Phi}_a F^{\Phi}_b. \tag{2.1}$$

Hereafter, we deal with dimensionless cluster operators which are normalized in such a way as to have a vacuum expectation value equal to one†

$$\langle 0|\Phi(0)|0\rangle = F^{\Phi}_0 = 1. \tag{2.2}$$

† Since the relevant primary operators will be identified with the cluster ones (except for their dimensional factors which can easily be restored), in the sequel we will adopt the same normalization for them as well.

In order to fully determine the FFs of the cluster operators we have chosen to focus on the set of all one- and two-particle FFs. Listing all the relations among them, one obtains a system of equations in the unknown parameters F_a^Φ and $a^{(k)}_{ab,\Phi}$. Let us see then all information we have on the FFs.

The first equations that one must consider are the *dynamical residue equations* resulting from the detailed analysis of the poles they are endowed with. These equations relate FFs with different external particles and may have a different origin. In particular, for every simple bound state pole of the amplitude S_{ab} at angle $\theta = \mathrm{i}u^c_{ab}$ relative to the particle A_c (see figure 1), we have

$$\lim_{\theta\to \mathrm{i}u^c_{ab}} (\theta - \mathrm{i}u^c_{ab}) F^\Phi_{ab}(\theta) = \mathrm{i}\Gamma^c_{ab} F^\Phi_c \tag{2.3}$$

where the on-mass-shell three-point coupling constant Γ^c_{ab} is given by the residue on the pole of the S-matrix

$$\lim_{\theta\to \mathrm{i}u^c_{ab}} (\theta - \mathrm{i}u^c_{ab}) S_{ab}(\theta) = \mathrm{i}(\Gamma^c_{ab})^2. \tag{2.4}$$

Dynamical residue equations are also provided by double-order poles and simple-order poles of type $\mathcal{B}$. Both of them are related to diagrams of the kind shown in figure 2. For each such diagram, one can write the following equation

$$\lim_{\theta_{ab}\to \mathrm{i}\varphi} (\theta_{ab} - \mathrm{i}\varphi) F^\Phi_{ab}(\theta_{ab}) = \mathrm{i}\Gamma^c_{ad}\Gamma^e_{db} F^\Phi_{ce}(\mathrm{i}\gamma) \tag{2.5}$$

where $\gamma = \pi - u^a_{cd} - u^b_{de}$. In the case of $\mathcal{B}$ poles one can always verify that the amplitude $S_{ce}(\theta)$ has a simple zero at $\theta = \mathrm{i}\gamma$. More complicated residue equations can, in general, be obtained with reference to $\mathcal{D}$ poles and higher-order ones whose explicit expressions—not reported here—can be however easily written, once the corresponding multiscattering diagrams have been identified.

It must be stressed that the above set of equations just depend on the dynamics of the model through its S-matrix and hold identical for every operator $\Phi(x)$. Therefore, in general, some residual freedom on the parameters is still expected after imposing these equations, because they must be satisfied by the FFs of all operators compatible with the assumed asymptotic behaviour.

Adding to this system of *linear* equations the *nonlinear* cluster equations (2.1) of the two-particle FFs, one obtains in general a redundant set of compatible equations in all the unknown parameters of the one- and two-particle FFs. Due to its nonlinearity, the system allows a multiplicity of solutions which define the so-called *cluster operators* of the theory†. If the number of solutions of the system matches the cardinality of the Kac table of the model one is led to identify them with the families of FFs of the relevant primaries.

Among the cluster solutions, one can first identify the FFs of the deforming field $\phi_{1,k}$. This operator is known to be essentially the trace of the energy–momentum tensor $\Theta(x)$ since

$$\Theta(x) = 4\pi \mathcal{E}_{\mathrm{vac}} \phi_{1,k} \tag{2.6}$$

$\mathcal{E}_{\mathrm{vac}}$ being the vacuum energy density which can be easily computed by TBA computations [21]

$$\mathcal{E}_{\mathrm{vac}} = -\frac{m_1^2}{8\sum_{x\in P_{11}} \sin(\pi x)}. \tag{2.7}$$

† In all cases analysed, the smallest system of equations among different FFs which is sufficient to determine their coefficients turns out to involve just a subset of the two-particle FFs. This suggests that also in the general case it should be possible to predict the final number of cluster solutions already from a 'minimal' system, avoiding in this way dealing with systems of equations involving a huge number of unknown variables.

Here the set P_{11} is defined in equation (A.1) and m_1 is the lightest particle mass. In view of proportionality (2.6), the FFs of $\phi_{1,k}$ can be identified among the cluster solutions by checking the peculiar equations which characterize the two-particle FFs of $\Theta(x)$ in virtue of the conservation of the energy–momentum tensor, namely the normalization of the diagonal two-particle FFs

$$F^{\Theta}_{aa}(\mathrm{i}\pi) = 2\pi m_a^2 \tag{2.8}$$

and the factorization of the polynomial Q^{Θ}_{ab} for non-diagonal two-particle FFs ($a \neq b$) into

$$Q^{\Theta}_{ab}(\cosh\theta) = (2m_a m_b \cosh\theta + m_a^2 + m_b^2) R^{\Theta}_{ab}(\cosh\theta) \tag{2.9}$$

where R^{Θ}_{ab} is a suitable polynomial [1, 3].

Knowing the FFs of $\Theta(x)$, one is then enabled to make use of sum rule (1.4) to compute the conformal dimension of the operators defined by the remaining cluster solutions in order to identify them with all the relevant primaries of the theory. This sum rule can be evaluated by using the spectral representation of the correlator

$$\langle\Theta(x)\phi(0)\rangle_c = \sum_{n=1}^{\infty}\sum_{a_i}\int_{\theta_1>\theta_2\ldots>\theta_n}\frac{\mathrm{d}^n\theta}{(2\pi)^n}F^{\Theta}_{a_1,\ldots,a_n}(\theta)F^{\phi}_{a_1,\ldots,a_n}(\mathrm{i}\pi-\theta)\mathrm{e}^{-|x|\sum_{k=1}^{n}m_k\cosh\theta_k}. \tag{2.10}$$

In all the models we have studied, the corresponding series for sum rule (1.4) displays a very fast convergence behaviour for any of the cluster operators. The truncated sums obtained by including just a few contributions have proved sufficient to attain a good approximation of all the values expected by the Kac table of conformal dimensions. In this way, the one-to-one correspondence between cluster solutions and primary relevant operators can easily be set.

Finally, having obtained the FFs of all the relevant fields in each integrable deformation, as a further check of their correct identification, one may employ formulae (1.3) relative to the universal ratios of the nearby non-integrable quantum field theories. These predictions can then be compared with their numerical estimates obtained from the truncated conformal space (TCS) approach developed in [13]. The agreement between numerical estimates and

Table 1. One-particle form factors of cluster solutions in $[\mathcal{M}(\frac{2}{7})]_{(1,2)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$
$F_1^{\mathcal{O}}$	0.812 944 7456i	1.245 503 611i
$F_2^{\mathcal{O}}$	−0.120 038 7686	−0.465 676 6285

Table 2. Two-particle form factors coefficients of cluster solutions in $[\mathcal{M}(\frac{2}{7})]_{(1,2)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$
$a^{(0)}_{11,\mathcal{O}}$	−0.690 535 5776	−0.417 821 7785
$a^{(1)}_{11,\mathcal{O}}$	1.570 496 171	3.686 419 944
$a^{(0)}_{12,\mathcal{O}}$	31.912 171 66i	160.820 065 8i
$a^{(1)}_{12,\mathcal{O}}$	25.763 371 82i	153.126 224 4i
$a^{(0)}_{22,\mathcal{O}}$	−12.748 049 09	−71.664 591 55
$a^{(1)}_{22,\mathcal{O}}$	−3.976 635 89	−59.846 898 51

Table 3. One-particle form factors of cluster solutions in $[\mathcal{M}(\frac{2}{7})]_{(1,3)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$
$F_1^{\mathcal{O}}$	0.870 338 7193i	1.408 237 641i
$F_2^{\mathcal{O}}$	−0.332 266 1173	−0.869 884 0033

Table 4. Two-particle form factors coefficients of cluster solutions in $[\mathcal{M}(\frac{2}{7})]_{(1,3)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$
$a^{(0)}_{11,\mathcal{O}}$	−1.453 085 043	−3.804 226 098
$a^{(0)}_{12,\mathcal{O}}$	10.389 248 46i	30.409 860 50i
$a^{(1)}_{12,\mathcal{O}}$	6.420 908 640i	27.199 406 17i
$a^{(0)}_{22,\mathcal{O}}$	−13.763 819 09	−42.189 514 12
$a^{(1)}_{22,\mathcal{O}}$	−4.702 281 947	−32.229 921 04

theoretical predictions of the non-integrable effects may provide additional confirmation and may remove all possible remaining doubts about the validity of the cluster hypothesis for these models.

3. Integrable deformations of $\mathcal{M}(2, 7)$

The minimal conformal model $\mathcal{M}(2, 7)$ has, in addition to the identity operator $\phi_{1,1}$, only two primary operators, $\phi_{1,2}$ and $\phi_{1,3}$, both of them relevant, the conformal weights being $-\frac{2}{7}$ and $-\frac{3}{7}$ respectively [4]. The perturbations of the conformal action either by the 'magnetic operator' $\phi_{1,2}$ or by the 'thermal operator' $\phi_{1,3}$ are both known to be, separately, integrable [18]. The S-matrices and the mass ratios of the two integrable models are given in tables B1 and B2. In their massive phase, both perturbations have two stable massive particles denoted by A_1 and A_2, with a mass ratio and a scattering matrix which depend on the integrable direction considered. In each case, we expect to find two non-trivial independent families of FF solutions to the cluster equations (1.6) (in addition to the family of the null FFs relative to the identity operator).

The FFs of the primary operators of the model relative to the thermal deformation have already been considered in [8]. Here, we have performed an *ab initio* calculation by imposing the cluster equations: our result has been in perfect agreement with the FFs of [8], proving in this way that these cluster solutions are also unique.

The result of the computation of FFs in the two integrable deformations $[\mathcal{M}(2, 7)]_{(1,2)}$ and $[\mathcal{M}(2, 7)]_{(1,3)}$ are summarized in tables 1–2 and 3–4, respectively, where we list the values of the one-particle FFs and the coefficients $a^{(k)}_{ab,\phi}$ of the two-particle FFs relative to some of the lightest two-particle states. As expected, we find two non-trivial solutions of FFs families. In each deformation, the FFs of the deforming operator suitably rescaled by (2.6), can be immediately identified because they satisfy the peculiar equations characterizing the trace of the energy–momentum tensor (2.8) and (2.9). This is further confirmed by employing the spectral representation of the correlator $\langle\Theta(x)\Theta(0)\rangle_c$ in sum rule (1.5), which provides in both deformations the value of the central charge with a very high precision (the relative error being of the order of 10^{-4}–10^{-5}). The identification of both the solutions with the primaries $\phi_{1,2}$ and $\phi_{1,3}$ is easily established after computing for each solution

Table 5. Sum rules of the conformal dimensions of primary operators in $[\mathcal{M}(\frac{2}{7})]_{(1,2)}$.

States	s	Δ_{12}-terms	Δ_{13}-terms
A_1	$1.000m_1$	−0.292 2910	−0.447 8157
A_2	$1.969m_1$	0.001 6428	0.006 3729
$A_1\ A_1$	$\geqslant 2.000m_1$	0.005 1123	0.013 7590
$A_1\ A_2$	$\geqslant 2.969m_1$	−0.000 0763	−0.000 4400
$A_1\ A_1\ A_1$	$\geqslant 3.000m_1$	−0.000 1040	−0.000 4777
$A_2\ A_2$	$\geqslant 3.939m_1$	0.000 0003	0.000 0040
Sum		−0.285 7159	−0.428 5976
Value expected		−0.285 7143	−0.428 5714

Table 6. Sum rules of the conformal dimensions of primary operators in $[\mathcal{M}(\frac{2}{7})]_{(1,3)}$.

States	s	Δ_{12}-terms	Δ_{13}-terms
A_1	$1.000m_1$	−0.322 1795	−0.521 2974
A_2	$1.618m_1$	0.029 0206	0.075 9768
$A_1\ A_1$	$2.000m_1$	0.009 8699	0.025 8398
$A_1\ A_2$	$2.618m_1$	−0.002 3149	−0.008 9996
$A_1\ A_1\ A_1$	$3.000m_1$	−0.000 3334	−0.001 3803
$A_2\ A_2$	$3.236m_1$	0.000 1155	0.000 6612
Sum		−0.285 8218	−0.429 1998
Value expected		−0.285 7143	−0.428 5714

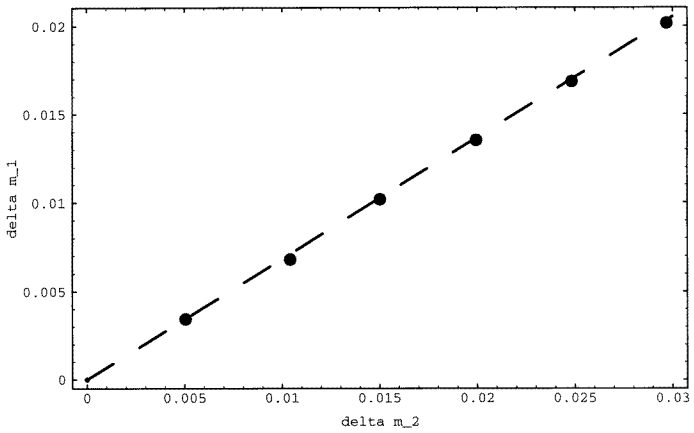

Figure 4. Numerical TCS estimates of δm_1 versus δm_2 for different values of the 'non-integrable' coupling in the model $[\mathcal{M}(\frac{2}{7})]_{(1,2)} + \varepsilon\phi_{1,3}$. The broken line represents the theoretical prediction.

its UV anomalous dimension by means of sum rule (1.4). The contributions to this sum rule coming from the dominant lightest multiparticle states are given in tables 5 and 6 for the two deformations (the contributions are ordered according to increasing values of the Mandelstam variable s of the multiparticle state). The agreement of the truncated sums with the known values of the anomalous dimensions is very satisfactory given the fast

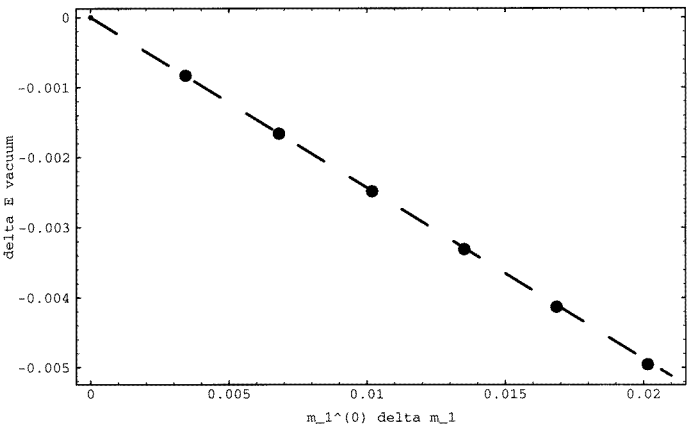

Figure 5. Numerical TCS estimates of $\delta\mathcal{E}_{\rm vac}$ versus $m_1^{(0)}\delta m_1$ for different values of the 'non-integrable' coupling in the model $[\mathcal{M}(\frac{2}{7})]_{(1,2)}+\varepsilon\phi_{1,3}$. The broken line represents the theoretical prediction.

Table 7. One-particle form factors of cluster solutions in $[\mathcal{M}(\frac{2}{9})]_{(1,2)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$F_1^{\mathcal{O}}$	0.754 830 171 7i	1.288 575 652i	1.564 862 744i
$F_2^{\mathcal{O}}$	−0.105 690 972 5	−0.459 339 8099	−0.733 160 9072
$F_3^{\mathcal{O}}$	−0.013 756 840 37i	−0.117 538 9994i	−0.285 481 7817i

Table 8. Two-particle form factors coefficients of cluster solutions in $[\mathcal{M}(\frac{2}{9})]_{(1,2)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$a^{(0)}_{11,\mathcal{O}}$	−0.381 024 8990	0.128 088 8115	0.644 962 9545
$a^{(1)}_{11,\mathcal{O}}$	1.289 925 788	3.759 118 917	5.543 942 595
$a^{(0)}_{12,\mathcal{O}}$	14.109 051 83i	75.186 320 19i	110.347 205 6i
$a^{(1)}_{12,\mathcal{O}}$	−12.743 237 79i	−79.908 954 89i	−180.984 509 2i
$a^{(2)}_{12,\mathcal{O}}$	−19.369 980 44i	−143.709 687 2i	−278.559 252 2i
$a^{(0)}_{13,\mathcal{O}}$	−1.826 322 080	−18.975 400 47	−51.567 863 33
$a^{(1)}_{13,\mathcal{O}}$	−1.116 015 559	−16.277 743 86	−48.012 790 71
$a^{(0)}_{22,\mathcal{O}}$	−1.466 545 085	−3.003 367 424	14.916 065 4
$a^{(1)}_{22,\mathcal{O}}$	7.821 352 950	60.495 406 24	160.400 770 5
$a^{(2)}_{22,\mathcal{O}}$	2.717 967 823	51.337 734 03	130.787 766 4
$a^{(0)}_{23,\mathcal{O}}$	153.827 946 7i	1842.946 063i	5426.663 81i
$a^{(1)}_{23,\mathcal{O}}$	175.558 426 8i	2962.508 857i	9796.436 391i
$a^{(2)}_{23,\mathcal{O}}$	30.431 247 86i	1130.002 086i	4380.673 323i
$a^{(0)}_{33,\mathcal{O}}$	−32.421 103 24	−450.093 615 5	−1394.808 207
$a^{(1)}_{33,\mathcal{O}}$	−20.232 937 66	−589.137 653 0	−2309.626 757
$a^{(2)}_{33,\mathcal{O}}$	−2.174 915 595	−158.770 199 3	−936.616 509 6

convergency behaviour of the spectral series. In the computation of these sum rules, some three-particle FF contributions have been inserted as well, although we do give here their

Table 9. One-particle form factors of cluster solutions in $[\mathcal{M}(\frac{2}{9})]_{(1,3)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$F_1^{\mathcal{O}}$	0.802 076 5716i	1.445 292 066i	1.802 249 672i
$F_2^{\mathcal{O}}$	−0.313 911 1339	−1.019 263 084	−1.584 911 324
$F_3^{\mathcal{O}}$	−0.137 369 2453i	−0.556 196 7434i	−1.002 231 818i

Table 10. Two-particle form factors coefficients of cluster solutions in $[\mathcal{M}(\frac{2}{9})]_{(1,3)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$a^{(0)}_{11,\mathcal{O}}$	−0.963 149 2344	−3.127 326 026	−4.862 860 736
$a^{(0)}_{12,\mathcal{O}}$	10.646 966 13i	40.739 514 64i	72.355 681 81i
$a^{(1)}_{12,\mathcal{O}}$	5.908 620 424i	34.570 483 56i	67.032 198 61i
$a^{(0)}_{13,\mathcal{O}}$	−2.592 348 236	−11.329 779 18	−21.249 129 75
$a^{(1)}_{13,\mathcal{O}}$	−1.153 703 500	−8.417 302 355	−18.913 504 58
$a^{(0)}_{22,\mathcal{O}}$	−5.978 990 567	−26.440 699 21	−49.876 748 76
$a^{(1)}_{22,\mathcal{O}}$	−1.544 771 430	−16.286 335 59	−39.378 641 16

exact expression for the sake of simplicity (their general parametrization follows the one adopted, for instance, in [3]). It should be noticed that the oscillating behaviour of these sums is typical of non-unitary theories where one expects, in general, both positive and negative terms.

3.1. Non-integrable deformations of $\mathcal{M}(2, 7)$

For each possible integrable deformation of the model, the addition of a further orthogonal deformation breaks its integrability leading, among other things, to corrections of the mass spectrum and of the vacuum energy. Both corrections can be independently computed by performing a numerical diagonalization of the off-critical Hamiltonian by means of the so-called truncation method [13]. We have carried out this analysis by comparing this non-integrable data with the theoretical predictions by equations (1.3). Let us briefly describe the output of these studies.

The double non-integrable deformation

$$[\mathcal{M}(2, 7)]_{(1,3)} + \varepsilon\phi_{1,2}$$

for small values of $\varepsilon m_1^{2\Delta_{1,2}-2}$ has already been studied in [9], where a good agreement between numerical and theoretical values has been found. Having obtained the FFs for the $\phi_{1,2}$ deformation, we are now able to complete the analysis by testing the opposite deformation

$$[\mathcal{M}(2, 7)]_{(1,2)} + \varepsilon\phi_{1,3}.$$

The numerical determination of the two universal ratios of equations (1.3) (for small values of $\varepsilon m_1^{2\Delta_{1,3}-2}$) gives $\frac{\delta m_1}{\delta m_2} = 0.675$ and $\frac{\delta \mathcal{E}_{\rm vac}}{\delta m_1} = -0.244 m_1^{(0)}$ with a precision estimated to be up to a few per cent. These values fully agree with the computed theoretical values $\frac{\delta m_1}{\delta m_2} = 0.684\,04$ and $\frac{\delta \mathcal{E}_{\rm vac}}{\delta m_1} = -0.243\,65 m_1^{(0)}$ (see, for instance, figures 4 and 5 where the data relative to the ratios $\frac{\delta m_1}{\delta m_2}$ and $\frac{\delta \mathcal{E}_{\rm vac}}{\delta m_1}$, respectively, are reported for different values of ε).

Table 11. One-particle form factors of cluster solutions in $[\mathcal{M}(\frac{2}{9})]_{(1,4)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$F_1^{\mathcal{O}}$	−0.904 354 489 8i	−1.727 853 39i	−2.211 259 663i
$F_2^{\mathcal{O}}$	−0.548 364 896 1	−1.476 188 315	−2.169 493 373
$F_3^{\mathcal{O}}$	0.267 331 650 8i	0.870 931 9528i	1.459 023 71i
$F_4^{\mathcal{O}}$	−0.084 881 189 64	−0.348 974 9771	−0.645 179 5597

Table 12. Two-particle form factors coefficients of cluster solutions in $[\mathcal{M}(\frac{2}{9})]_{(1,4)}$.

$\mathcal{O}$	$\phi_{1,2}$	$\phi_{1,3}$	$\phi_{1,4}$
$a^{(0)}_{11,\mathcal{O}}$	1.623 982 681	4.256 426 530	6.219 867 507
$a^{(1)}_{11,\mathcal{O}}$	−0.977 841 1563	−1.325 966 569	−1.477 684 504
$a^{(2)}_{11,\mathcal{O}}$	−2.029 027 259	−7.406 691 607	−12.130 814 76
$a^{(0)}_{12,\mathcal{O}}$	−9.935 037 127i	−30.489 350 00i	−50.024 039 46i
$a^{(1)}_{12,\mathcal{O}}$	−4.790 105 254i	−24.636 860 52i	−46.337 733 09i
$a^{(0)}_{13,\mathcal{O}}$	−45.210 741 97	−145.960 073 0	−220.960 971 0
$a^{(1)}_{13,\mathcal{O}}$	−441.375 608 6	−1583.189 947	−2731.357 697
$a^{(2)}_{13,\mathcal{O}}$	−533.323 714 0	−2301.408 527	−4364.089 257
$a^{(3)}_{13,\mathcal{O}}$	−139.417 354 0	−867.798 450 5	−1860.500 753
$a^{(0)}_{14,\mathcal{O}}$	44.340 329 61i	189.607 734 8i	357.835 776 2i
$a^{(1)}_{14,\mathcal{O}}$	54.791 940 08i	275.950 767 5i	562.963 880 1i
$a^{(2)}_{14,\mathcal{O}}$	11.433 958 82i	89.814 745 54i	212.503 853 9i
$a^{(0)}_{22,\mathcal{O}}$	−9.190 266 093	−30.910 940 92	−52.205 345 46
$a^{(1)}_{22,\mathcal{O}}$	−2.709 639 668	−19.636 124 53	−42.412 015 02
$a^{(0)}_{23,\mathcal{O}}$	−81.758 024 20i	−304.224 483 8i	−530.887 240 9i
$a^{(1)}_{23,\mathcal{O}}$	−92.611 281 43i	−446.860 116 9i	−884.972 303 4i
$a^{(2)}_{23,\mathcal{O}}$	−16.665 339 65i	−146.157 172 0i	−359.844 459 9i

4. Integrable deformations of $\mathcal{M}(2, 9)$

In this section, we turn our attention to the $\mathcal{M}(2, 9)$ minimal model which displays a richer structure in the RG space of relevant couplings. This model has in fact, besides the identity, three primary operators $\phi_{1,2}$, $\phi_{1,3}$ and $\phi_{1,4}$ which are all relevant with conformal dimensions $-\frac{1}{3}$, $-\frac{5}{9}$ and $-\frac{2}{3}$ respectively. These fields taken separately give rise to different integrable deformations of the conformal model, each of them characterized by a different mass spectrum and S-matrix (see tables B3, B4 and B5 in appendix B). In particular, the first two deformations produce three-particle mass spectra (with different mass ratios) while the last one gives a four-particle spectrum.

The FFs of the primary operators in the $\phi_{1,3}$-deformation have already been obtained in [8] and are known to satisfy the cluster property. Again, our derivation of these FFs as solutions of the cluster equations proves that the FFs found in [8] are the only possible cluster solutions.

The FFs of the cluster solutions for each of the three above-mentioned deformations have been computed according to the strategy explained in section 2. The resulting one-particle FFs and two-particle FFs coefficients are given in tables 7–8, 9–10 and 11–12 respectively. The important result is that in each integrable deformation of this model, three families of

Table 13. Sum rules of the conformal dimensions of primary operators in $[\mathcal{M}(\frac{2}{5})]_{(1,2)}$.

States	s	Δ_{12}-terms	Δ_{13}-terms	Δ_{14}-terms
A_1	$1.000m_1$	−0.340 9847	−0.582 0972	−0.706 9063
A_2	$1.982m_1$	0.001 7003	0.007 3894	0.011 7945
$A_1\ A_1$	$\geqslant 2.000m_1$	0.006 1957	0.020 7909	0.031 6698
A_3	$2.931m_1$	−0.000 0132	−0.000 1126	−0.000 2734
$A_1\ A_2$	$\geqslant 2.982m_1$	−0.000 0951	−0.000 7084	−0.001 4392
$A_1\ A_1\ A_1$	$\geqslant 3.000m_1$	−0.000 1421	−0.000 9038	−0.001 7386
$A_1\ A_3$	$\geqslant 3.931m_1$	0.000 0009	0.000 0117	0.000 0339
$A_2\ A_2$	$\geqslant 3.965m_1$	0.000 0004	0.000 0061	0.000 0157
$A_2\ A_3$	$\geqslant 4.914m_1$	−0.000 0000	−0.000 0002	−0.000 0008
Sum		−0.333 3379	−0.555 6241	−0.666 8445
Value expected		−0.333 3333	−0.555 5556	−0.666 6667

Table 14. Sum rules of the conformal dimensions of primary operators in $[\mathcal{M}(\frac{2}{9})]_{(1,3)}$.

States	s	Δ_{12}-terms	Δ_{13}-terms	Δ_{14}-terms
A_1	$1.000m_1$	−0.370 679	−0.667 941	−0.832 909
A_2	$1.802m_1$	0.031 509	0.102 310	0.159 088
$A_1\ A_1$	$\geqslant 2.000m_1$	0.013 898	0.045 127	0.070 170
A_3	$2.247m_1$	−0.004 839	−0.019 592	−0.035 304
$A_1\ A_2$	$\geqslant 2.802m_1$	−0.003 604	−0.018 722	−0.035 573
$A_1\ A_1\ A_1$	$\geqslant 3.000m_1$	−0.000 628	−0.003 514	−0.006 763
$A_1\ A_3$	$\geqslant 3.247m_1$	0.000 663	0.004 114	0.008 844
$A_2\ A_2$	$\geqslant 3.604m_1$	0.000 211	0.001 684	0.003 864
Sum		−0.333 469	−0.556 534	−0.668 583
Value expected		−0.333 333	−0.555 556	−0.666 667

Table 15. Sum rules of the conformal dimensions of primary operators in $[\mathcal{M}(\frac{2}{5})]_{(1,4)}$.

States	s	Δ_{12}-terms	Δ_{13}-terms	Δ_{14}-terms
A_1	$1.000m_1$	−0.451 081	−0.861 833	−1.102 950
A_2	$1.486m_1$	0.121 478	0.327 017	0.480 603
A_3	$1.956m_1$	−0.022 989	−0.074 895	−0.125 468
$A_1\ A_1$	$\geqslant 2.000m_1$	0.035 896	0.121 577	0.197 637
$A_1\ A_2$	$\geqslant 2.486m_1$	−0.023 279	−0.101 138	−0.183 618
A_4	$2.827m_1$	0.001 546	0.006 354	0.011 748
$A_1\ A_3$	$\geqslant 2.956m_1$	0.004 304	0.022 374	0.045 474
$A_2\ A_2$	$\geqslant 2.973m_1$	−0.001 535	−0.009 929	−0.022 429
$A_2\ A_3$	$\geqslant 3.443m_1$	−0.000 330	−0.002 101	−0.004 686
$A_1\ A_4$	$\geqslant 3.827m_1$	0.003 595	0.020 054	0.040 870
Sum		−0.332 396	−0.552 519	−0.662 819
Value expected		−0.333 333	−0.555 556	−0.666 667

non-trivial solutions have been found. Among the solutions, we have first identified the FFs of the deforming field by checking the exact fulfilment of equations (2.8) and (2.9), after

Table 16. Comparison between numerical and theoretical estimates of data obtained in different non-integrable deformations of $\mathcal{M}_{(2,9)}$.

	$\frac{\delta m_1}{\delta m_2}$		$\frac{\delta \mathcal{E}_{\rm vac}}{m_1^{(0)}\delta m_1}$	
Deformation	Numerical (±3%)	Theoretical	Numerical (±3%)	Theoretical
$[\mathcal{M}(2,9)]_{(1,2)} + \varepsilon\phi_{1,3}$	0.590	0.592 049	−0.275	−0.275 404
$[\mathcal{M}(2,9)]_{(1,2)} + \varepsilon\phi_{1,4}$	0.661	0.660 963	−0.204	−0.204 124
$[\mathcal{M}(2,9)]_{(1,3)} + \varepsilon\phi_{1,2}$	0.390	0.391 396	−1.04	−1.038 26
$[\mathcal{M}(2,9)]_{(1,3)} + \varepsilon\phi_{1,4}$	0.811	0.836 81	−0.205	−0.205 640
$[\mathcal{M}(2,9)]_{(1,4)} + \varepsilon\phi_{1,2}$	−0.133	−0.131 367	1.73	1.745 82
$[\mathcal{M}(2,9)]_{(1,4)} + \varepsilon\phi_{1,3}$	0.238	0.240 486	−0.550	−0.548 156

the appropriate rescaling (2.6). Moreover, the c sum rule (1.5) can be easily shown to give very precise approximations of the central charge in each of the three separate deformations.

As for the other solutions, they have been successfully identified with the FFs of the primary operators by computing their anomalous dimension by means of equation (1.4). The first contributions to these sums are given in tables 13–15. In all cases the agreement with the expected anomalous dimensions of the primaries is established, even though the convergence of the series is observed to be noticeably faster for lower absolute values of the anomalous dimension of the deforming field. This observed trend is indeed expected from the short-distance behaviour of the correlator (2.10), as predicted by the operator product expansion of the fields. In fact, in the models $\mathcal{M}(2, 2n+1)$ where the fields have negative anomalous dimensions, this correlator displays a zero at the origin whose power-law exponent is larger for lower absolute values of the anomalous dimension of $\Theta(x)$; correspondingly, the small x region of integration in (1.4) is less relevant making the lightest multiparticle states more dominant in the series.

4.1. Non-integrable deformations of $\mathcal{M}(2, 9)$

The availability of the FFs of all the primary fields of the model has allowed us, in each of the three separate integrable deformations, to consider two different orthogonal non-integrable deformations. We have then had the possibility of testing the theoretical values obtained for the universal quantities (1.3) against their numerical TCS estimates in six different multiple deformations, exploring in this way the non-integrable region around the conformal point of the model. The outcome of the analysis in all the deformations is summarized in table 16. Since the precision of TCS data is expected to be of approximately a few per cent, the comparison with the computed theoretical values is in all cases quite satisfactory.

5. Conclusions

The main purpose of this work has been to substantiate by means of concrete *ab initio* calculations the cluster hypothesis for the FFs of the relevant operators in integrable quantum field theories obtained as deformation of a conformal action. We have studied, in particular, the matrix elements of the primary operators in the integrable deformations of the first models of the non-unitary series $\mathcal{M}(2, 2n+1)$. In all cases analysed, we have confirmed the cluster hypothesis since we have found a one-to-one correspondence between the independent solutions of the cluster equations and the relevant fields.

It should be said that the absence of internal symmetries of the above models has played an important role in carrying out our computations. In fact, in this situation one can exploit the cluster equations (1.6) in their full generality. It would be interesting to see how the results of [10] generalize to the case of quantum field theories with internal symmetries which induce selection rules on the matrix elements. Another important open problem is also to understand the meaning of the cluster properties in quantum field theories which cannot be regarded as deformation of conformal models. A complete understanding of all these aspects of the FFs would allow us to improve our understanding of the asymptotic high-energy regime of quantum theories and their operator content.

Acknowledgments

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Appendix A

In this appendix we give the general parametrization adopted throughout the paper for two-particle FFs. The S-matrices of the specific models analysed in this paper are given in appendix B where the generic amplitude

$$S_{ab}(\theta) = \prod_{x \in P_{ab}} (x)^{p_x} \prod_{y \in Z_{ab}} (-y)^{q_y} \tag{A.1}$$

is written adopting the notation

$$(\alpha) = \frac{\tanh \frac{1}{2}(\theta + \mathrm{i}\pi\alpha)}{\tanh \frac{1}{2}(\theta - \mathrm{i}\pi\alpha)} \tag{A.2}$$

and the positive rational indices x and y label the poles and the zeros displayed by the amplitude in the physical strip $\mathrm{Im}\ \theta \in [0, \pi]$. The bound state simple poles of the S-matrices are identified by superscripts which denote the particles produced. There are however also simple poles of a different nature which have been labelled with suffices $\mathcal{B}$ and $\mathcal{D}$ that are not related to any bound state and are due to multiparticle scattering processes of the kind shown in figures 2 and 3 respectively. The fact that these diagrams (which usually produce second- and third-order poles) are here responsible for simple poles is due to the occurrence of zeros in the S-matrix factors carried by the internal crossing lines [20]. Higher-order poles are present as well and, among these, in the model $[\mathcal{M}(\frac{2}{9})]_{(1,4)}$, some triple poles labelled by an asterisk which also have a non-standard diagrammatic interpretation. The understanding of the nature of all the poles is necessary in order to assign the correct pole structure to the FFs.

The general two-particle FF of a scalar operator $\Phi(x)$

$$F^{\Phi}_{ab}(\theta_1 - \theta_2) = \langle 0|\Phi(0)|A_a(\theta_1)A_b(\theta_2)\rangle \tag{A.3}$$

will be parametrized by

$$F^{\Phi}_{ab}(\theta) = Q^{\Phi}_{ab}(\theta)\frac{F^{\min}_{ab}(\theta)}{D_{ab}(\theta)}. \tag{A.4}$$

where the 'minimal' FF

$$F^{\min}_{ab}(\theta) = (-\mathrm{i}\sinh(\theta/2))^{\frac{1-S_{ab}(0)}{2}} \frac{\prod_{x \in P} g_x^{p_x}(\theta)}{\prod_{y \in Z_{ab}} g_y^{q_y}(\theta)} \tag{A.5}$$

which has neither zeros nor poles in the physical strip, is written in terms of the function

$$g_x(\theta) = \prod_{k=0}^{\infty} \left[\frac{\left[1 + \left[\frac{i\pi-\theta}{2\pi}\right]^2\right]\left[1 + \left[\frac{i\pi-\theta}{2\pi}\right]^2\right]}{\left[1 + \left[\frac{i\pi-\theta}{2\pi}\right]^2\right]\left[1 + \left[\frac{i\pi-\theta}{2\pi}\right]^2\right]} \right]^{k+1} . \tag{A.6}$$

This function is normalized by $g_x(i\pi) = 1$ and behaves asymptotically as

$$g_x(\theta) \sim e^{|\theta|/2} \qquad \text{for } \theta \to \infty. \tag{A.7}$$

The factor $Q^{\Phi}_{ab}(\theta)$ in (A.4) is a polynomial in $\cosh\theta$ carrying the dependence on the specific operator $\Phi(x)$

$$Q^{\Phi}_{ab} = \sum_{k=0}^{k^{\max}_{ab,\Phi}} a^{(k)}_{ab,\Phi} \cosh^k(\theta). \tag{A.8}$$

The most subtle element in the parametrization of the FFs is represented by the structure of the poles which, in equation (A.4) are introduced by the factor $D_{ab}(\theta)$. In order to establish which poles are to be found in a FF one must in general have a complete understanding of the nature of the poles in the corresponding S-matrix element in terms of microscopical processes. We will write in general,

$$D_{ab}(\theta) = \prod_{x \in P_{ab}} \mathcal{P}_x^{i_x} \mathcal{P}_{1-x}^{j_x} \tag{A.9}$$

where the set of indices is defined in (A.1) and

$$\mathcal{P}_x(\theta) = \frac{\cos(\pi x) - \cosh(\theta)}{2\cos^2(\frac{\pi x}{2})}. \tag{A.10}$$

For bound-state simple poles and ordinary higher-order poles of the S-matrix, the correct rule for determining the indices i_x and j_x is given by [3]

$$\begin{aligned} &i_x = n \qquad j_x = n-1 \qquad \text{if } p_x = 2n-1 \\ &i_x = n \qquad j_x = n \qquad \text{if } p_x = 2n. \end{aligned} \tag{A.11}$$

For simple poles of type $(x)_{\mathcal{B}}$ and $(x)_{\mathcal{D}}$ one can show that the correct indices are still $i_x = 1$ and $j_x = 0$, as for a bound state simple pole.

Notice, however, that the poles of the FFs induced by the triple poles labelled with $*$ in $[\mathcal{M}(\frac{2}{9})]_{(1,4)}$ do not fall within the above analysis. Their general expressions are not investigated further here since these FFs were not needed in the present work.

As a final remark, notice that every function (α) could be equivalently written as $(1-\alpha)$ without changing the S-matrices. However, the pole prescription given above for the FFs is *sensitive* to this change in the case of odd-order poles. Therefore, all the labels α in the S-matrices reported here have been chosen to give (in units of $i\pi$) the value of the direct s-channel resonant angles in the case of bound state odd poles and also in the case of poles of type $\mathcal{B}$ and $\mathcal{D}$†. Only with this choice, the above prescription gives the correct poles of the FFs.

With parametrization (A.4), the two-particle FF of a general operator Φ is therefore completely determined after fixing the coefficients $a^{(k)}_{ab,\Phi}$ in expansion (A.8).

† For s-channel in these cases we mean the one defined by figures 2 and 3 with particles flowing upwards.

Appendix B

In this appendix we give the S-matrices of the integrable models analysed in this work. The function (α) used in the tables is given in equation (A.2). Anomalous simple poles have been labelled with $\mathcal{B}$ and $\mathcal{D}$, while the anomalous triple poles of the model $[\mathcal{M}(\frac{2}{9})]_{(1,4)}$ are identified with $*$.

Table B1. S-matrix and mass ratios of the $[\mathcal{M}(\frac{2}{7})]_{(1,2)}$ model.

$$S_{11}(\theta) = (\overset{\mathbf{1}}{\tfrac{2}{3}})\ (\overset{\mathbf{2}}{\tfrac{1}{9}})\ (-\tfrac{2}{9})$$

$$S_{12}(\theta) = (\overset{\mathbf{1}}{\tfrac{17}{18}})\ (\tfrac{11}{18})_{\mathcal{B}}$$

$$S_{22}(\theta) = (\overset{\mathbf{2}}{\tfrac{2}{3}})\ (\tfrac{8}{9})_{\mathcal{B}}\ (\tfrac{5}{9})_{\mathcal{D}}$$

$$m_2 = 2\cos\frac{\pi}{18} m_1 = 1.9696\ldots m_1$$

Table B2. S-matrix and mass ratios of the $[\mathcal{M}(\frac{2}{7})]_{(1,3)}$ model.

$$S_{11}(\theta) = (\overset{\mathbf{2}}{\tfrac{2}{5}})$$

$$S_{12}(\theta) = (\overset{\mathbf{1}}{\tfrac{4}{5}})\ (\overset{\mathbf{2}}{\tfrac{3}{5}})$$

$$S_{22}(\theta) = (\overset{\mathbf{1}}{\tfrac{4}{5}})\ (\tfrac{2}{5})^2$$

$$m_2 = 2\cos\frac{\pi}{5} m_1 = 1.618\,03\ldots m_1$$

Table B3. S-matrix and mass ratios of the $[\mathcal{M}(\frac{2}{9})]_{(1,2)}$ model.

$$S_{11}(\theta) = (\overset{\mathbf{1}}{\tfrac{2}{3}})\ (\overset{\mathbf{2}}{\tfrac{1}{12}})\ (-\tfrac{1}{4})$$

$$S_{12}(\theta) = (\overset{\mathbf{1}}{\tfrac{23}{24}})\ (\overset{\mathbf{3}}{\tfrac{1}{8}})\ (\tfrac{5}{8})_{\mathcal{B}}\ (-\tfrac{5}{24})$$

$$S_{13}(\theta) = (\overset{\mathbf{2}}{\tfrac{11}{12}})\ (\tfrac{7}{12})_{\mathcal{B}}$$

$$S_{22}(\theta) = (\overset{\mathbf{2}}{\tfrac{2}{3}})\ (\tfrac{11}{12})^2\ (\tfrac{7}{12})_{\mathcal{D}}\ (-\tfrac{1}{4})$$

$$S_{23}(\theta) = (\overset{\mathbf{1}}{\tfrac{23}{24}})\ (\tfrac{7}{8})_{\mathcal{B}}\ (\tfrac{5}{8})_{\mathcal{B}}\ (\tfrac{13}{24})_{\mathcal{D}}$$

$$S_{33}(\theta) = (\overset{\mathbf{3}}{\tfrac{2}{3}})\ (\tfrac{11}{12})_{\mathcal{B}}\ (\tfrac{5}{6})_{\mathcal{B}}\ (\tfrac{7}{12})_{\mathcal{D}}\ (\tfrac{1}{2})_{\mathcal{D}}$$

$$m_a = \frac{\sin\frac{a\pi}{24}}{\sin\frac{\pi}{24}} m_1 \qquad a = 1, 2, 3$$

Table B4. S-matrix and mass ratios of the $[\mathcal{M}(\frac{2}{9})]_{(1,3)}$ model.

$$S_{11}(\theta) = (\overset{\mathbf{2}}{\tfrac{2}{7}})$$

$$S_{12}(\theta) = (\overset{\mathbf{1}}{\tfrac{6}{7}})\ (\overset{\mathbf{3}}{\tfrac{3}{7}})$$

$$S_{13}(\theta) = (\overset{\mathbf{2}}{\tfrac{5}{7}})\ (\overset{\mathbf{3}}{\tfrac{4}{7}})$$

$$S_{22}(\theta) = (\overset{\mathbf{3}}{\tfrac{4}{7}})\ (\tfrac{5}{7})^2$$

$$S_{23}(\theta) = (\overset{\mathbf{1}}{\tfrac{6}{7}})\ (\overset{\mathbf{2}}{\tfrac{5}{7}})\ (\tfrac{3}{7})^2$$

$$S_{33}(\theta) = (\overset{\mathbf{1}}{\tfrac{6}{7}})\ (\tfrac{3}{7})^2\ (\tfrac{5}{7})^2$$

$$m_a = \frac{\sin\frac{a\pi}{7}}{\sin\frac{\pi}{7}} m_1 \qquad a = 1, 2, 3$$

Table B5. S-matrix and mass ratios of the $[\mathcal{M}(\frac{2}{9})]_{(1,4)}$ model.

$$S_{11}(\theta) = (\overset{\mathbf{1}}{\tfrac{2}{3}})\ (\overset{\mathbf{2}}{\tfrac{7}{15}})\ (\overset{\mathbf{3}}{\tfrac{2}{15}})\ (-\tfrac{1}{15})\ (-\tfrac{2}{5})$$

$$S_{12}(\theta) = (\overset{\mathbf{1}}{\tfrac{23}{30}})\ (\overset{\mathbf{3}}{\tfrac{13}{30}})$$

$$S_{13}(\theta) = (\overset{\mathbf{1}}{\tfrac{14}{15}})\ (\overset{\mathbf{2}}{\tfrac{11}{15}})\ (\overset{\mathbf{4}}{\tfrac{1}{5}})\ (\tfrac{3}{5})^2\ (-\tfrac{2}{15})\ (-\tfrac{1}{3})$$

$$S_{14}(\theta) = (\overset{\mathbf{3}}{\tfrac{13}{15}})\ (\tfrac{8}{15})_{\mathcal{B}}\ (\tfrac{2}{3})^2$$

$$S_{22}(\theta) = (\overset{\mathbf{2}}{\tfrac{2}{3}})\ (\overset{\mathbf{4}}{\tfrac{1}{5}})\ (\tfrac{8}{15})_{\mathcal{B}}$$

$$S_{23}(\theta) = (\overset{\mathbf{1}}{\tfrac{5}{6}})\ (\tfrac{1}{2})_{\mathcal{B}}\ (\tfrac{7}{10})_{\mathcal{B}}\ (\tfrac{11}{30})_{\mathcal{B}}$$

$$S_{24}(\theta) = (\overset{\mathbf{2}}{\tfrac{9}{10}})\ (\tfrac{23}{30})_{\mathcal{B}}\ (\tfrac{3}{10})_{\mathcal{B}}\ (\tfrac{19}{30})_{\mathcal{B}}\ (\tfrac{17}{30})^2$$

$$S_{33}(\theta) = (\overset{\mathbf{3}}{\tfrac{2}{3}})^3\ (\tfrac{2}{15})^2\ (\tfrac{7}{15})^2\ (-\tfrac{1}{15})\ (-\tfrac{2}{5})$$

$$S_{34}(\theta) = (\overset{\mathbf{1}}{\tfrac{14}{15}})\ (\tfrac{4}{5})_{\mathcal{B}}\ (\tfrac{7}{15})_{\mathcal{D}}\ (\tfrac{11}{15})^2\ (\tfrac{3}{5})^3_*$$

$$S_{44}(\theta) = (\overset{\mathbf{4}}{\tfrac{2}{3}})^3_*\ (\tfrac{8}{15})^3_*\ (\tfrac{2}{5})_{\mathcal{D}}\ (\tfrac{11}{15})_{\mathcal{B}}\ (\tfrac{13}{15})_{\mathcal{B}}\ (\tfrac{1}{5})^2$$

$$m_2 = 2\cos\frac{7\pi}{30} m_1 = 1.486\,29\ldots m_1$$

$$m_3 = 2\cos\frac{\pi}{15} m_1 = 1.956\,30\ldots m_1$$

$$m_4 = 2\cos\frac{\pi}{10} m_2 = 2.827\,09\ldots m_1$$

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